10/16/24

Representation Theory for finite W-algebras

(Notes & Speaker: Nikolay)

Goals: Skryabin thm, Exactness & Monoidality of Whittaker Reduction functor, HC-bimod for W, category $\mathcal{O}$ for W-alg.

Refs: [GG] Gan-Ginzburg : Quantization of Slodowy slice
[Gin] Ginzburg : HC-bimod for W-alg.

Setup

- $\mathfrak{g}$ = semisimple Lie alg $/\mathbb{C}$, Killing form $(\,,\,): \mathfrak{g}\times\mathfrak{g}^* \to \mathbb{C}$
- $\mathbb{O}\subset\mathfrak{g}$ a nilpotent orbit. Pick $e\in\mathbb{O}$ & let $\chi\in\mathfrak{g}^*$ such that $(e,\chi)=1$.
  "Whittaker character"
- Pick $\mathfrak{sl}_2$-triple $\langle e,h,f\rangle \longrightarrow \mathfrak{g}$ by Jacobson-Morozov
- $S = e+\ker \operatorname{ad} f$ = Slodowy slice to $\mathbb{O}$ at e

Key fact: S may be obtained via Hamiltonian reduction:

The $\operatorname{ad} h$-action on $\mathfrak{g}$ $\Rightarrow$ $\mathfrak{g}=\bigoplus_{i\in\mathbb{Z}} \mathfrak{g}(i)$, $\mathfrak{g}(i)=\{x\in\mathfrak{g} \mid [h,x]=ix\}$

Pick Lagrangian $\mathfrak{l}\subseteq\mathfrak{g}(-1)$ (w.r.t. $x,y\longmapsto\chi([x,y])$), and define nilpotent subalgebra $\mathfrak{n}:=\mathfrak{l}\oplus\left(\bigoplus_{i\le -2}\mathfrak{g}(i)\right)$. Note, $\chi([\mathfrak{n},\mathfrak{n}])=0$.

(We omit $\mathfrak{l}$ from notation from now on.)

$N:=\exp(\mathfrak{n})$, is unipotent subgroup of dimension $=\frac{1}{2}\dim\mathbb{O}$.

$\mu:\mathfrak{g}^*\longrightarrow\mathfrak{n}^*$ $\rightsquigarrow$ $\boxed{S=\mu^{-1}(\chi)/\!\!/N \quad \text{"Whittaker" Hamiltonian reduction,}}$

$\Rightarrow \mathbb{C}[S]=\mathbb{C}[\mu^{-1}(\chi)]^N$ is Poisson algebra.

(If $\mathbb{O}$ principal, then $\mathbb{C}[S]\simeq\mathbb{C}[\mathfrak{g}]^G$).

Quantizing $\Rightarrow$ $W:=\Big(\underbrace{U\mathfrak{g}/U\mathfrak{g}\,(x-\chi(x))_{x\in\mathfrak{n}}}_{Q}\Big)^{N\text{ left}}$ is associative algebra, called the W-alg. associated to $(\mathfrak{g},e)$.

(Omit $\mathfrak{l}$ from notation).

$\mathfrak{n}_\chi := (x-\chi(x))_{x\in\mathfrak{n}}$

Have alg isom $\operatorname{End}_{U\mathfrak{g}}(Q)^{op} \underset{h\longmapsto h(1\otimes 1)}{\simeq} W = \{v\in Q : xv=\chi(x)v \ \ \forall x\in\mathfrak{n}\}$

Recall <u>Kazhdan grading</u>: Introduce modified $\mathbb{C}^*$-action on $\mathfrak{g}^*$ so that it stabilizes S & contracts it to e: $\forall\ \xi \in \mathfrak{g}^*$

Jacobson-Morozov

$$t.\xi := t^{-2}\, Ad^*(\gamma(t))(\xi), \text{ where } \gamma: t \longmapsto \underset{\mathfrak{sl}_2}{\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}} \to G.$$

This induces $\underbrace{\mathbb{C}[\mathfrak{g}^*](n)}_{\text{kazhdan grading}} := \{F \in \mathbb{C}[\mathfrak{g}^*] \mid t.F = t^n F\ \forall t \in \mathbb{C}^*\}$

& similarly, get kazhdan grading on $\mathbb{C}[S]$ (non-negative)

$$U_n\mathfrak{g}(i) = \{x \in U_n\mathfrak{g} \mid (\mathrm{ad}\, h)(x) = ix\}$$

PBW

$\rightsquigarrow$ kazhdan filtration $F_n U\mathfrak{g} = \{U_j\mathfrak{g}(i) \mid i+2j \leq n\}$

$\implies$ Have Kazhdan filtration on W such that

$$gr(W) \simeq \mathbb{C}[S] \quad \underline{\text{isom of graded Poisson algebras,}}$$

equipped w/ Kazhdan grading

<u>slogan:</u> quantization commutes w/ reduction

§ Skryabin

Let $(U\mathfrak{g}, \mathfrak{n}_\chi)$-mod = category of fin gen'd $U\mathfrak{g}$-mod w/ $\mathfrak{n}_\chi = (x-\chi(x))_{x\in\mathfrak{n}}$ acting locally

<u>= Whittaker category.</u>

<u>Ex</u> $Q := U\mathfrak{g} / U\mathfrak{g}\, \mathfrak{n}_\chi \in (U\mathfrak{g}, \mathfrak{n}_\chi)$-mod.

for "$\mathfrak{n}_\chi$-invariants = χ-twisted $\mathfrak{n}$-invariants."

<u>Def</u> $Wh^{\mathfrak{n}}(E) = \{x \in E \mid (n-\chi(n)).x = 0\ \forall n \in \mathfrak{n}\}$

<u>Thm</u> (Skryabin) $\{\text{fin. gen } W\text{-mod}\} \underset{\longleftarrow}{\xrightarrow{V \longmapsto Q\otimes_W V}} (U\mathfrak{g}, \mathfrak{n}_\chi)\text{-mod}$ is equivalence of categories, $Wh^{\mathfrak{n}}(E) \longleftarrow\!\shortmid E$

<u>Rmk:</u> This is analogues to Kashiwara's Thm: D-modules on Y being equiv. to D-mod on X supported on $Y \subseteq X$.

Pf Observe $Wh^n(E)=0 \implies E=0$.

Let $V \in W\text{-mod}$ be fixed. It has a Kazhdan filtration coming from the one on W. ($F_i V = F_i W \cdot V_0$, V_0 = finite generating set of V).

The isom $N \times S \xrightarrow{\sim} \mu^{-1}(\chi)$ induces

$$\mathbb{C}[N] \otimes gr(W) \simeq gr(Q) \quad \text{isom of graded algebras} \quad (*)$$

Thus $H^0(\mathfrak{n}, gr(Q) \otimes_{gr(W)} gr(V)) = (\mathbb{C}[N] \otimes gr(V)) = gr(V)$

H^0 = $\mathfrak{n}$-invariants

& $H^i(\mathfrak{n}, gr(Q) \otimes_{gr(W)} gr(V)) = 0 \quad \forall i > 0$

$(*) \Rightarrow gr(Q)$ free over $gr(W) \implies gr(Q \otimes_W V) \simeq gr Q \otimes_{gr(W)} gr(V)$

Then, a spectral sequence argument yields

$$H^0(\mathfrak{n}, Q \otimes_W V) = V \;\; \& \;\; H^i(\mathfrak{n}, Q \otimes_W V) = 0 \quad \forall i > 0.$$

H^0 = "$\mathfrak{n}_\chi$ - invariants"!

Thus, $\underline{Wh^n(Q \otimes_W V) = V}^{(+)}$ & remains $F: Q \otimes_W Wh^n(E) \xrightarrow{\sim} E$. $(**)$

Let $E' = \ker F$, $E'' = \operatorname{coker} F$.

Then $Wh^n(E') = E' \cap Wh^n(Q \otimes_W Wh(E)) \overset{(+)}{=\!=} E' \cap Wh^n(E) = 0$

$\implies E' = 0 \implies F$ injective

And, $0 \to Q \otimes_W Wh^n(E) \to E \to E'' \to 0$ induces LES

$$0 \to \underset{\overset{(+)\,\shortparallel}{Wh^n(E)}}{H^0(\mathfrak{m}, Q \otimes_W Wh^n(E))} \xrightarrow[\text{by def}]{\sim} H^0(\mathfrak{m}, E) \to \underbrace{H^0(\mathfrak{m}, E'')}_{\text{Thus equals } 0.} \to \underbrace{H^1(\mathfrak{m}, Q \otimes_W Wh^n(E))}_{\overset{\shortparallel}{0}\;(+)}$$

$\Rightarrow H^0(\mathfrak{n}, E'') = Wh^{\mathfrak{n}}(E'') = 0 \Rightarrow E'' = 0$ ∎

D-mod interpretation:

Def An M-equivariant D-module on G/B, $\mathcal{V}$, is $\mathfrak{n}$-Whittaker with respect to $\chi : \mathfrak{n} \to \mathbb{C}$ if for any $x \in \mathfrak{n}$ & $v \in \mathcal{V}$,

$$(x_D - x_M).v = \chi(x) \cdot v$$

where x_D = vector field corresponding to x acting on $\mathcal{V}$

x_M = action obtained by differentiating the M-equivariant action.

Next, the projection $U_{\mathfrak{g}} \to Q$ maps $Z_{\mathfrak{g}} \hookrightarrow W$ injectively.

Let $Z_+ \subseteq Z_{\mathfrak{g}}$ be augmentation ideal

Thm (BB localization + Skryabin)

$$\{W/Z_+W\text{ - modules}\} \simeq \left\{ \begin{matrix} \mathfrak{n}\text{-Whittaker coherent D-modules} \\ \text{on } G/B \quad (\text{w.r.t. } \chi : \mathfrak{n} \to \mathbb{C}) \end{matrix} \right\}$$ ∎

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§ Whittaker Reduction

Def Given a $(U_{\mathfrak{g}}, U_{\mathfrak{g}})$-bimod K, define Whittaker reduction functor

$$Wh^{\mathfrak{n}}_{\mathfrak{n}}(K) := (K/K\mathfrak{n}_\chi)^{\mathfrak{n}}, \text{ where } \mathfrak{n}_\chi = (x - \chi(x))_{x \in \mathfrak{n}}$$

χ-twisted $\mathfrak{n}$-invariants

$\mathfrak{n}_\chi$-coinvariants

& for $n \in \mathfrak{n}$, $x \in K/K\mathfrak{n}_\chi$, $n.x := \operatorname{ad} n(x) + \chi(n).x$

Let $\mathcal{K}(K) := Wh^{\mathfrak{n}}_{\mathfrak{n}}(K)$.

We may write $\mathcal{K}(K) = \operatorname{Hom}_{U_{\mathfrak{g}}}(Q, K \otimes_{U_{\mathfrak{g}}} Q) \rightsquigarrow$ is (W,W)-bimodule.

Recall: $\mathcal{K}(U_{\mathfrak{g}}) = W \simeq Z_{\mathfrak{g}}$ for principal e / nondegen. χ.

Prop K is a monoidal functor : U_g-bimod $\longrightarrow$ W-bimod.

Pf (Gannon–Ginzburg). We must show for any U_g-bimods M, M', that $K(M) \otimes_W K(M') = K(M \otimes_{U_g} M')$.

By Skryabin, $Q \otimes_W K(M) \simeq M \otimes_{U_g} Q$. (Follows from taking $(\ast\flat) \otimes_{U_g} Q$)

Thus, $\left[\cancel{(Q \otimes_W} K(M)) \otimes_W K(M')\right]^N \xrightarrow{\sim} (M \otimes_{U_g} Q) \otimes_W K(M')$ (Skryabin)

$\xrightarrow{\sim} M \otimes_{U_g} (Q \otimes_W K(M'))$

$\xrightarrow{\sim} \left[(M \otimes_{U_g} M') \cancel{\otimes_{U_g} Q}\right]^N$ (Skryabin)

Then apply $N^{\exp(n)}$-invariants & use that $(Q \otimes_W -)^N$ & $(- \otimes_{U_g} Q)^N$ are both identity functors! ∎

Rmk This awesome proof holds for HC-bimodules in much greater generality!

Break

Next, we show $K(-)$ is an exact functor, but for this we'll need to restrict to Harish-Chandra bimodules.

Def A finitely generated (U_g, U_g)-bimod K for which the adjoint g-action $\mathrm{ad}\, a: v \longmapsto av - va$ is locally finite is called a Harish-Chandra bimodule

Def M coherent D-module, has good filtration $F_\bullet$ & define characteristic variety $\quad Ch(\mathcal{M}) = \mathrm{Supp}(\mathrm{gr}^F \mathcal{M})$.

K a HC $U_{\mathfrak{g}}$ bimod $\Longrightarrow$ $Ch(K) \subseteq \Delta \subseteq \mathrm{gr}(U_{\mathfrak{g}} \times U_{\mathfrak{g}}) = \mathbb{C}[\mathfrak{g}^*] \otimes \mathbb{C}[\mathfrak{g}^*]$

Def Given a pair (B, B') of non-neg. filtered algebras s.t. $\mathrm{gr} B \cong \mathrm{gr} B'$ and are fin. gen, then say a (B,B')-bimod K is weak Harish-Chandra bimodule if its characteristic variety $Ch(K) \subseteq \mathrm{Spec}(\mathrm{gr} B) \times \mathrm{Spec}(\mathrm{gr} B')$ is contained in the diagonal $\Delta \subseteq \mathrm{Spec}(\mathrm{gr} B) \times \mathrm{Spec}(\mathrm{gr} B')$.

Thm Let $\mu, \nu \in \mathfrak{h}^*$ be dominant regular weights & K a weak-HC-(U_μ, U_ν)-bimod. Then

central reduction

BB-localization

Def 1.1 of Kashiwara-Kawai

(1) K is a HC-(U_μ, U_ν)-bimod $\Longleftrightarrow$

(2) the associated D-mod $Loc(K)$ on $G/B \times G/B$ has regular singularities $\Longleftrightarrow$

(3) There exists a good filtration on $Loc(K)$ whose associated graded is reduced.

This motivates def. of HC W-bimod: Let $\pi: T^*G/B \to \mathfrak{g}^*$ Springer resolution & let $\tilde{S} := \pi^{-1}(S) =$ "Slodowy variety"

$$\boxed{\text{Def: A weak HC-}(W_c, W_{c'})\text{-bimod } K \text{ is HC if the corresponding D-mod } Loc(K) \text{ on } \tilde{S} \times \tilde{S} \text{ has a good filtration whose corresponding associated graded is reduced.}}$$

Rmk • More precisely, Ginzburg defines an intermediate object, called directed algebra $\mathcal{U}(e, \mathcal{B}) \otimes \mathcal{U}(e, \gamma)$ so that $Loc(K) \in \mathcal{U}(e, \mathcal{B}) \otimes \mathcal{U}(e, \gamma)$-gr mod, & then requires that to admit a good filtration so that $\mathrm{gr}(Loc\, K) \in Coh(\tilde{S} \times \tilde{S})$ is reduced.

• See §6 for further discussions.

Thm (Main Thm 4.1.4 of [Gin]) Let $c, c' \in \operatorname{Spec} Z_{\mathfrak{g}}$ & let

$$U_c = U_{\mathfrak{g}} / U_{\mathfrak{g}} \cdot (x - c(x))_{x \in Z_{\mathfrak{g}}} \quad \& \quad W_c := W / W \cdot i(x - c(x))_{x \in Z_{\mathfrak{g}}}, \text{ where } Z_{\mathfrak{g}} \xhookrightarrow{i} W.$$

are central reductions.

(1) Then $K \longmapsto Wh^m_m(K)$ induces a faithful exact functor

$$\frac{\{HC\text{-}(U_c, U_{c'})\text{-bimod}\}}{\left\{\begin{array}{l} HC\text{-}(U_c, U_{c'})\text{-bimod supported} \\ \text{on } \mathcal{N}^{\diamond} = \bigcup\limits_{\operatorname{Ad} G\text{-orbits } \mathbb{O}' : \mathbb{O} \not\subset \bar{\mathbb{O}}'} \mathbb{O}' \end{array}\right\}} \xrightarrow{K \mapsto Wh^m_m(K')} \{HC\text{-}(W_c, W_{c'})\text{-bimod}\}$$

(2) For $K \in HC\text{-}(U_c, U_{c'})$-bimod, the characteristic variety $Ch(K)$ satisfies:

$$\mathbb{O} \subseteq Ch(K) \iff Wh^m_m K \neq 0$$

$$\bar{\mathbb{O}} = Ch(K) \iff \dim(Wh^m_m K) < \infty.$$

Thus: Whittaker reduction only remembers the simplest part

Pf sketch The faithfullness is observed by constructing a right adjoint

$$\underset{HC\text{-}(W_c, W_{c'})\text{-bimod}}{N} \longmapsto \operatorname{Hom}^{fin}_{W_c}(Q_c, Q_{c'} \otimes_{W_{c'}} N) \in HC\text{-}(U_c, U_c')\text{-bimod}.$$

fin = ad g- locally finite part

Lets explain exactness in quasiclassical case: Let $\chi : \mathfrak{n} \to \mathbb{C}$ non degen Whittaker character

Let $\underbrace{T^{\chi}}_{\text{sometimes denoted } T^{*,\chi}(G/N)} = T^*(G) /\!\!/_{\chi} N$ be Kostant-Whittaker reduction of T^*G along (N, χ).

Then T^{χ} is affine w/ $\mathcal{O}(T^{\chi}) = \mathcal{O}(G) \overset{\mathcal{O}(N)}{\otimes} \mathcal{O}(\mathbb{A}^d)$ for some d.

Slodowy slice

Then, $\quad gr(K) : (\mathrm{Sym}(\mathfrak{g})\text{-mod})^G \longrightarrow Z_{\mathfrak{g}}\text{-mod}$

$$M \longmapsto \left(\mathcal{O}(T^\chi) \otimes_{\mathrm{Sym}\,\mathfrak{g}} M\right)^G$$

Since G reductive, suffices $\mathcal{O}(T^\chi)$ is flat $\mathrm{Sym}\,\mathfrak{g}$-module. But this follows from the map $T^\chi \longrightarrow \mathfrak{g}^*$ being smooth [Ginzburg-Kazhdan §3], which consequently implies pullback is flat. ✓

Now, the general exactness is proved in [Gin] §4 by a spectral sequence argument. Some comments:

- One obstacle in using the Kazhdan filtration on W-mods is that it is a priori unbounded from below, so associated graded is infinite-dimensional. This explains why work with HC (dg-bimod.).
- Also, the spectral sequence does not converge in 1st quadrant, so Ginzburg introduces 2 good filtrations to compute the spectral sequences & consequently show they converge to the quasi-classical limit.
- $HC(U_c, U_c) \xrightarrow{wh_n} (U_c, n_\chi)\text{-mod} \xrightarrow{wh^n} W_c\text{-mod}$

  exact by above spectral seq. argument

  Equivalence by Skryabin.

  $K \longmapsto K/K n_\chi \qquad M \longmapsto M^{n_\chi}$

<u>Application</u>: Proof of Premet's conjecture that relates finite-dimensional W-modules to primitive ideals $I \subseteq U\mathfrak{g}$ such that the associated variety of I equals $\overline{\mathrm{Ad}\,G(e)}$. (Losev gave alternate proof using deformation quantization).

Remark: Raskin gives geometric proof of exactness of Whittaker reduction which we'll see when discussing the affine story. Roughly speaking, HC-(U_g, U_g)-bimod are G-equivariant coherent sheaves on g^*, & the Whittaker reduction is given by restricting to $S = e + \ker(\operatorname{ad} f)$ & then taking N-invariants. The G-equivariance implies (in principal case) that restricting to S is same as restricting to g^{reg}, but open restriction is exact! Also, N-acts freely so taking N-invariants also exact (cf. Skryabin) ☒

§ Category $\mathcal{O}$ discussions

Let $Q = Z_G(e,h,f)$. There's an embedding $q = Lie(Q) \hookrightarrow W$ & pick Cartan $t \subseteq q$ & set $l = z_g(t)$. Then l = minimal Levi containing e. Pick $\theta \in t$ such that $z_g(\theta) = l$ (category considered depends on θ).

Def Category $\mathcal{O}(\theta)$ consists of W-modules N such that:

(1) N is fin gen'd

(2) $t \subseteq W$ acts diagonally on N

(3) $W_{>0} := \bigoplus_{i>0} \{w \in W : [\theta, w] = iw\}$ acts locally nilpotently on N.

(4) $N^{W_{>0}} := \{m : w.m = 0 \ \forall w \in W_{>0}\}$ is finite dimensional

Ex 1) When e = principal nilpotent, then $q = \{0\}$, $l = g$, $\theta = 0$, and conditions 2, 3 are vacuous so $\mathcal{O}(\theta) = \{\text{Finite.dim } W\text{-mod}\}$

Facts • Verma modules $\Delta^\theta(N^0) := W \otimes_{W_{\geq 0}} N^0$ where

N^0 is $W_{\geq 0}/W_{\geq 0}^+$ -module, $W_{\geq 0}^+ = W_{\geq 0} \cap W \cdot W_{>0}$,
which is $\mathfrak{t}$ diagonalizable

- N^0 irreducible $\Rightarrow \exists!\ L^\theta(N^0)$ simple subquotient
- Cat $\mathcal{O}$ is Artinian (all objects fin. length) because all weight spaces are fin. dim & there are finitely many simples.

Let $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ be $\operatorname{ad}\theta$-eigenspaces, $\mathfrak{l} = \mathfrak{g}_0$, & form $\mathfrak{m}_0 \subseteq \mathfrak{g}_0$ analogues to before. This produces $W^0 = W(\mathfrak{g}_0, e)$.

Set $\tilde{\mathfrak{m}} = \mathfrak{m}_0 \oplus \mathfrak{g}_{>0} \subseteq \mathfrak{g}$ & $\tilde{\mathfrak{m}}_\chi = \{ \xi - \langle \chi, \xi \rangle : \xi \in \tilde{\mathfrak{m}} \}$

<u>Def</u> A $U\mathfrak{g}$-module M is <u>generalized Whittaker</u> (for e, θ) if:

1) M is fin gen'd
2) $\mathfrak{t}$ acts diagonally on M
3) $\tilde{\mathfrak{m}}_\chi$ acts locally nilpotently on M.

(*) Denote the resulting category by $Whit(\theta)$.

<u>Ex</u> $\Delta^{e,\theta}(N^0) := U\mathfrak{g} \otimes_{U(\mathfrak{g}_{\geq 0})} Sk_0(N^0) \in Whit(\theta)$ are the Vermas, where $Sk_0 : W^0\text{-mod} \longrightarrow U(\mathfrak{g}_0)\text{-mod}$ is the Skryabin functor for $(\mathfrak{g}_0, e)$.

<u>Thm</u> [Losev] There's an isom $\varphi : W^0 \xrightarrow{\sim} W_{\geq 0}/W_{\geq 0}^+$ & an equivalence

$$K : Whit(\theta) \longrightarrow \mathcal{O}(\theta).$$

Moreover, $K(\Delta^{e,\theta}(M)) \simeq \Delta^\theta(\varphi_* M)$. ← Two notions of Vermas coincide.

Remarks about <u>Fin. dim W-modules</u>:

- The naive expectation about having Borel-Weil-Bott theory is completely false.